Journal of Applied Mechanics and Technical Physics, Vol. 50, No. 4, pp. 677–683, 2009

LIMITING VELOCITY OF CRACK PROPAGATION IN ELASTIC MATERIALS

N. I. Chekunaev and A. M. Kaplan

UDC 539.3; 539.4

A crack is represented as a continuous set of linear dislocations. Simple analytical expressions are obtained for the potential and kinetic energies of the environment of moving cracks and the attached mass of cracks for an arbitrary form of the stress applied to the crack $P(x)$. It is shown that the indicated analytical expressions are bilinear integrals of the functions $P(x)$ and $\partial P(x)/\partial x$. These integrals are calculated in explicit form for a constant stress over the entire crack length and the stress due to the action of molecular adhesion forces in a narrow region near the crack openings. It is shown that the calculation results does not depend on the form of molecular adhesion forces. The proposed approach to describing cracks and calculations based on it has made it possible for the first time to obtain a complete analytical expression for the limiting crack propagation velocity in elastic materials as a function of the main mechanical characteristics of such materials.

Key words: *crack of critical size, limiting crack velocity, elastic body, attached mass, mechanical stress, dislocation.*

Introduction. In studies of the problem of cracking of solid materials, one of the most important problems is the determination (theoretical and experimental) of the dependence of the propagation velocity v of cracks on the main mechanical characteristics of the materials tested. According to available data, this dependence for homogeneous stressed materials was first derived theoretically by Mott [1]. Subsequent studies have shown that the dependence of crack propagation velocity on crack length found in [1] is incorrect [2, 3]. However, the important finding of [1] is that there exists the limiting velocity $v_{\max} = k\sqrt{E/\rho}$ for propagation of cracks of supercritical sizes in stressed materials as the crack size tends to infinity. The dimensionless coefficient k was not determined. This important parameter was calculated in [2]. Assuming that strain propagation in the vicinity of a moving crack in elastic materials is the same as that in the vicinity of a stationary motionless crack, Roberts and Wells [2] obtained a value $k = 0.38$ for only one value of the coefficient $\nu = 0.25$. At present, there have been a considerable number of papers analyzing various models for the propagation of crack of supercritical sizes [4–10]. However, in none of the papers was the practically important complete analytical expression obtained for the limiting velocity of crack propagation $v_{\max}$ in elastic materials as a function of the main mechanical characteristics of such materials, in particular, Poisson's ratio. The present paper is devoted to the derivation of this dependence.

1. Potential Energy of the Environment of a Loaded Crack. We consider a linear crack (Fig. 1) which has unit length in the direction normal to the plane of the figure and length $2a$ in the x direction. The y axis is perpendicular to the plane of the crack. An external stress $p(x)$ is applied to the surface of the crack. According to [11], a narrow region of size $d + \chi$ near the crack openings in the direction opposite to the direction of $p(x)$ is acted upon by molecular adhesion forces producing an additional stress $G(x)$ (see Fig. 1). Thus, the complete stress on the surface of the crack is given by

$$P(x) = p(x) - G(x).$$

We will assume that, at infinite distance from the crack, the stress is equal to zero. A linear crack in an elastic medium can be considered as a discontinuity line on which the normal displacement component has a jump:

Semenov Institute of Chemical Physics, Russian Academy of Sciences, Moscow 119991; nichek@mail.ru.
Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 50, No. 4, pp. 158–166, July–August, 2009. Original article submitted May 29, 2008.

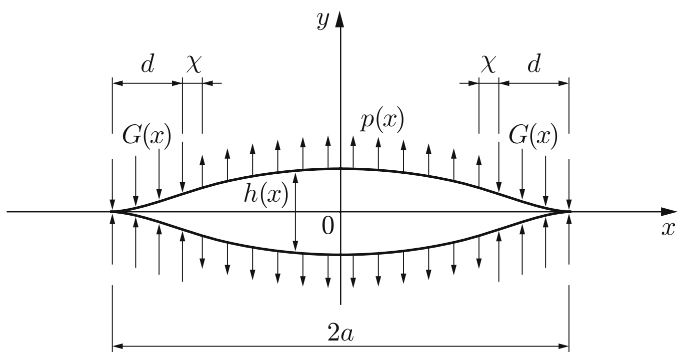

Fig. 1. Diagram of a linear crack of length $2a$.

$u_y(x) = \pm h(x)/2$ [12], where $h(x)$ is the variable crack width (see Fig. 1). According to [12], the crack can formally be treated as a set of rectilinear dislocations extended along the z axis and continuously distributed over the x axis in the interval $(-a, a)$ with Burgers vector $\boldsymbol{b} = (0, b, 0)$. The linear density of such dislocations will be denoted as $\rho(x)$. The crack width $h(x)$ is related to the quantity $\rho(x)$ by the formula [12]

$$h(x) = \int_{-a}^{x} \rho(x)\, dx.$$

In this case, the crack-induced strain u_i and stress σ_{ik} (dependent only on the coordinates x and y) are given by the expressions

$$u_i(x, y) = \int_{-a}^{a} u_i^d(x - \xi, y)\rho(\xi)\, d\xi; \tag{1.1}$$

$$\sigma_{ik}(x, y) = \int_{-a}^{a} \sigma_{ik}^d(x - \xi, y)\rho(\xi)\, d\xi,$$

where u_i^d and σ_{ik}^d are the strain and stress, respectively, produced by unit dislocation. Using the expressions given in [12] for u_i^d and σ_{ik}^d for plane strain, we express the strain in the material as

$$u_x(x, y) = -\frac{1}{4\pi(1-\nu)} \int_{-a}^{a} \Big(\frac{1-2\nu}{2} \ln\,[(x-\xi)^2 + y^2] + \frac{y^2}{(x-\xi)^2 + y^2}\Big)\rho(\xi)\, d\xi,$$

$$u_y(x, y) = \frac{1}{2\pi} \int_{-a}^{a} \Big(\arctan\frac{x-\xi}{y} + \frac{1}{2(1-\nu)}\,\frac{(x-\xi)y}{(x-\xi)^2 + y^2}\Big)\rho(\xi)\, d\xi. \tag{1.2}$$

From this, we find the following expressions for the stress tensor components for plane strain:

$$\sigma_{xx}(x, y) = \frac{E}{4\pi(1-\nu^2)} \int_{-a}^{a} \rho(\xi)\, \frac{(x-\xi)(y^2 - (x-\xi)^2)}{[y^2 + (x-\xi)^2]^2}\, d\xi,$$

$$\sigma_{yy}(x, y) = -\frac{E}{4\pi(1-\nu^2)} \int_{-a}^{a} \rho(\xi)\, \frac{(x-\xi)(3y^2 + (x-\xi)^2)}{[y^2 + (x-\xi)^2]^2}\, d\xi, \tag{1.3}$$

$$\sigma_{xy}(x, y) = \frac{E}{4\pi(1-\nu^2)} \int_{-a}^{a} \rho(\xi)\, \frac{y[y^2 - (x-\xi)^2]}{[y^2 + (x-\xi)^2]^2}\, d\xi.$$

Here and below, the integrals containing a pole are understood in the sense of the principal value, E is Young's modulus, and ν is Poisson's ratio. In the case of a crack with free ends, the dislocation density is given by the expression [12]

$$\rho(x) = -\frac{4(1-\nu^2)}{\pi E}\sqrt{a^2-x^2}\int_{-a}^{a}\frac{P(\xi)}{\sqrt{a^2-\xi^2}\,(\xi-x)}\,d\xi. \tag{1.4}$$

Disturbance of the equilibrium leads to a change in the crack size and, hence, in the stress distribution in the vicinity of the crack. Along with the molecular forces acting near the crack openings, for which exact expressions are not known, the propagation of a nonequilibrium crack gives rise to additional forces, including inertial forces, which can be taken into account by changing the form of the function $G(x)$. In real materials, in addition, forces arise due to the presence of a plastic region near the crack tips. Generally, the can lead to a dependence of the function $G(x)$ on the crack size and propagation velocity. Subsequently, by the quantity $G(x)$ is meant the stress due to molecular forces and the additional forces produced by the environment. It should be noted that if the molecular forces $G(x)$ are applied in a narrow (compared to the crack size) region near the crack openings, the final expressions do not depend on the concrete form of these forces. In this case, the natural assumption is made that the region adjacent to the crack is in a state of quasistatic equilibrium. In other words, the equilibrium conditions [12]

$$\int_{-a}^{a}\frac{P(x)\,dx}{\sqrt{a^2-x^2}} = 0, \tag{1.5}$$

which is valid for a static crack, is also satisfied for the examined case of motion of a nonequilibrium crack. For an equilibrium crack, this condition ensures that the stress vanishes at infinity [12]. In [10], a similar condition of quasistatic crack propagation is given in the form of equality of the dynamic coefficient of overstress and the dynamic specific energy of fracture, which can depend on the crack velocity and size.

Using expressions (1.3), it is possible to calculate the elastic energy stored in the material around a crack. The specific elastic energy f obeys the expression [12]

$$f = \frac{E}{2(1+\nu)}\Big(u_{ik}^2 + \frac{\nu}{1-2\nu}u_{ll}^2\Big), \tag{1.6}$$

where the strain tensor u_{ik} is related to the stress tensor by the formula

$$u_{ik} = [(1+\nu)\sigma_{ik} - \nu\sigma_{ll}\delta_{ik}]/E. \tag{1.7}$$

Here δ is the Kronecker symbol ($\delta_{ik} = 1$ at $i = k$, $\delta_{ik} = 0$ at $i \neq k$); the summation is performed over repeated indices.

An expression for the elastic energy U of the material stress in the vicinity of a crack can be obtained by substituting (1.3) into (1.6), (1.7) and integrating over the two-dimensional volume over x and y:

$$U = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy.$$

The volume integrals are calculated by standard methods. As a result, after some simplifications, we obtain the following expression for the elastic energy of the material:

$$U = -\frac{E}{8\pi(1-\nu^2)}\int_{-a}^{a}\int_{-a}^{a}\rho(\xi)\rho(\xi')\ln|\xi-\xi'|\,d\xi\,d\xi'. \tag{1.8}$$

Substitution of the expression for $\rho(x)$ from (1.4) into (1.8) yields

$$U = \frac{2(1-\nu^2)}{\pi E}\int_{-a}^{a}\int_{-a}^{a}P(\xi)P(\xi')J(\xi,\xi')\,d\xi\,d\xi',$$

where

$$J(x,y) = \ln\Big|\frac{\sqrt{1-x/a}\,/\sqrt{1+x/a} + \sqrt{1-y/a}\,/\sqrt{1+y/a}}{\sqrt{1-x/a}\,/\sqrt{1+x/a} - \sqrt{1-y/a}\,/\sqrt{1+y/a}}\Big|.$$

Taking into account the symmetry of the stress applied to the crack: $P(x) = P(-x)$, the expression for the elastic energy can be written as

$$U = \frac{4}{\pi}\,\frac{1-\nu^2}{E} \int_0^a \int_0^a \ln \left| \frac{\sqrt{1-\xi^2/a^2} + \sqrt{1-\xi'^2/a^2}}{\sqrt{1-\xi^2/a^2} - \sqrt{1-\xi'^2/a^2}} \right| P(\xi) P(\xi')\, d\xi\, d\xi'. \tag{1.9}$$

Using the formula [13]

$$\ln \frac{|q|}{|p|} = \int_0^\infty \frac{\cos pt - \cos qt}{t}\, dt,$$

expression (1.9) can be brought to the form

$$U = \frac{8}{\pi}\,\frac{1-\nu^2}{E}\, a^2 \int_0^\infty \frac{I_P^2(a,t)}{t}\, dt,$$

where

$$I_P(a,t) = \int_0^1 \sin\left(\sqrt{1-x^2}\, t\right) P(ax)\, dx.$$

Let us calculate the elastic energy for the case where a constant stress is applied to the crack: $p(x) = p_0$ and the crack-restraining stress due to molecular forces near the tip has the form of a smoothed step of width d. In this case, the size of the transition region χ is determined from the formula

$$G(x) = \frac{p_1}{2}\Big(1 + \tanh \frac{|x| - L}{\chi}\Big), \tag{1.10}$$

where $L = a - d$. It should be taken into account that the stress p_1 due to molecular forces is much higher than the external stress applied to the crack surface $p_1 \gg p_0$. For $\chi \to 0$, calculation of the integral in (1.5) yields

$$\arccos\,(L/a) = \pi p_0/(2p_1). \tag{1.11}$$

Calculation of the integral in (1.9) leads to the following expression for the elastic energy:

$$U = \frac{\pi(1-\nu^2)}{E}\, a^2 \Big\{ p_0^2 + \frac{4}{\pi}\, p_0 p_1 \Big[\frac{L}{a}\sqrt{1 - \frac{L^2}{a^2}} - \arccos\Big(\frac{L}{a}\Big)\Big]$$

$$+ \frac{4}{\pi^2}\, p_1^2 \Big[\arccos^2\Big(\frac{L}{a}\Big) - 2\,\frac{L}{a}\sqrt{1 - \frac{L^2}{a^2}}\, \arccos\Big(\frac{L}{a}\Big) - \frac{L^2}{a^2}\, \ln\Big(\frac{L^2}{a^2}\Big)\Big]\Big\}. \tag{1.12}$$

Substitution of (1.11) into (1.12) yields

$$U = -\frac{4(1-\nu^2)}{\pi E}\, a^2 p_1^2 \cos^2\Big(\frac{\pi}{2}\,\frac{p_0}{p_1}\Big) \ln\Big[\cos^2\Big(\frac{\pi}{2}\,\frac{p_0}{p_1}\Big)\Big].$$

For $p_1 \gg p_0$, we obtain the well-known expression

$$U = \pi(1-\nu^2) p_0^2 a^2/E,$$

which depends only on the applied external stress p_0 and does not depend on the form of the molecular forces.

2. Kinetic Energy of the Environment of a Moving Crack. We assume that the moving crack is in a quasi-equilibrium state, i.e., the stress around the crack is the same as that around the equilibrium crack. This assumption is valid if the crack propagates at a velocity much lower than the sound velocity in the material. The validity of this assumption is discussed below.

Local displacements in the vicinity of the crack are given by expressions (1.1) and (1.2). These displacement vary in time with variation in the crack size a. Then, for the propagating crack, local velocity components v_x and v_y are equal to

$$v_{x(y)}(x,y) = \frac{du_{x(y)}(x,y,a)}{dt} = \frac{\partial}{\partial a}\, u_{x(y)}(x,y,a)\, \frac{da}{dt} = \frac{\partial}{\partial a}\, u_{x(y)}(x,y,a)\, v.$$

Here $v = da/dt$ is the crack propagation velocity. Since $\rho(a, a) = \rho(-a, a) = 0$ [12], we can write

$$\frac{\partial}{\partial a}\, u_{x(y)}(x, y, a) = \int\limits_{-a}^{a} u^d_{x(y)}(x-\xi, y)\, \frac{\partial}{\partial a}\, \rho(\xi, a)\, d\xi.$$

It is easy to show that, if the equality (1.5) is satisfied, the following relation is valid:

$$\frac{\partial}{\partial a}\, \rho(x, a) = -\frac{4(1-\nu^2)}{\pi E}\, \frac{1}{\sqrt{a^2 - x^2}} \int\limits_{-a}^{a} \frac{\sqrt{a^2-\xi^2}}{\xi - x + i\delta}\, \frac{\partial}{\partial a}\, P(\xi, a)\, d\xi.$$

The total kinetic energy T of the material is equal to

$$T = \frac{\rho}{2} \int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} (v_x^2 + v_y^2)\, dx\, dy = \frac{\rho v^2}{2} \int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} dx\, dy \Big[\Big(\frac{\partial u_x(x,y,a)}{\partial a}\Big)^2 + \Big(\frac{\partial u_y(x,y,a)}{\partial a}\Big)^2\Big]$$

$$= \frac{\rho v^2}{2} \int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} dx\, dy \Big(\int\limits_{-a}^{a}\int\limits_{-a}^{a} \Big[u^d_x(x-\xi, y) u^d_x(x-\xi', y) + u^d_y(x-\xi, y) u^d_y(x-\xi', y)\Big] \frac{\partial}{\partial a}\, \rho(\xi, a)\, \frac{\partial}{\partial a}\, \rho(\xi', a)\, d\xi\, d\xi'\Big) \tag{2.1}$$

(ρ is the density of the material). We introduce the attached (associated) mass of the crack m by the relation

$$T = mv^2/2. \tag{2.2}$$

Calculation of integrals (2.1) using expressions (1.2) yields

$$m(a) = \int\limits_{-a}^{a}\int\limits_{-a}^{a} \Theta(\xi, \xi')\, \frac{\partial}{\partial a}\, P(\xi, a)\, \frac{\partial}{\partial a}\, P(\xi', a)\, d\xi\, d\xi', \tag{2.3}$$

where

$$\Theta(x, y) = \frac{4\rho(1+\nu)^2}{\pi E^2} \Big\{\Big(\nu^2 - \frac{3}{2}\,\nu + \frac{7}{8}\Big)\Big[(x-y)^2 \ln\Big|\frac{\sqrt{(a-x)/(a+x)} - \sqrt{(a-y)/(a+y)}}{\sqrt{(a-x)/(a+x)} + \sqrt{(a-y)/(a+y)}}\Big|$$
$$-\, 2\Big(1 - \ln\frac{2R}{a}\Big)\sqrt{a^2-x^2}\sqrt{a^2-y^2}\,\Big] + \Big(3\nu^2 - \frac{21}{4}\,\nu + \frac{45}{16}\Big)\sqrt{a^2-x^2}\sqrt{a^2-y^2}$$
$$+\, \frac{\sqrt{a^2-x^2}\sqrt{a^2-y^2}}{12R^2}\Big[\Big(-3\nu^2 + \frac{9}{2}\,\nu - \frac{33}{8}\Big)xy + x^2 + y^2 + a^2\Big]\Big\}. \tag{2.4}$$

In (2.1), the integrals have a logarithmic singularity, because of which the truncation radius R is introduced. The physical meaning of the radius R is the size of the region of the material involved in the crack propagation, which is determined below. To calculate the attached mass of the crack by formula (2.3), we can assume that, for small χ, the function G in (1.10) is a step function. Then,

$$\frac{\partial P(x,a)}{\partial a} = p_1[\delta(x-L) + \delta(x+L)]\, \frac{\partial L}{\partial a}, \tag{2.5}$$

where $\delta(x)$ is a Dirac function. Substituting (2.4) and (2.5) into (2.3) and retaining the first terms of the series in powers of p_0/p_1, we obtain the following expression for the attached mass of the crack:

$$m(a) = 8\pi\, \frac{p_0^2}{E^2}\, \rho a^2 (1+\nu)^2 \Big[\Big(\nu^2 - \frac{3}{2}\,\nu + \frac{7}{8}\Big) \ln\Big(\frac{2R}{a}\Big) - \frac{3}{8}\,\nu + \frac{3}{32} + \frac{a^2}{8R^2}\Big]. \tag{2.6}$$

Thus, the obtained expression does not depend on the form of the molecular forces.

3. Crack Propagation Velocity in Elastic Materials. The crack propagation velocity can be found by using the energy conservation law, which is written as

$$\frac{d}{dt}\,(T + U + E_S - W) = 0. \tag{3.1}$$

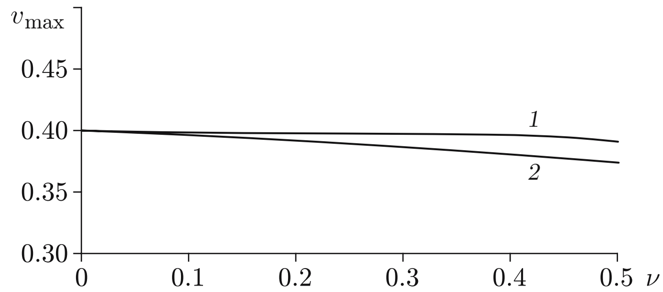

Fig. 2. Limiting velocity v_{max} versus Poisson's ratio ν: 1) plane strain; 2) plane stress state.

Here $E_S = 4\Gamma a$ is the surface energy of the crack, Γ is the surface tension coefficient of the material, W is the work of external forces [3]:

$$W = \int_{-a}^{a} h(x)P(x)\,dx = 2F. \tag{3.2}$$

According to relations (1.13), (2.2), and (3.2), from (3.1) we obtain

$$\frac{d}{dt}\Big(\frac{m(a)v^2}{2} - \frac{\pi(1-\nu^2)p_0^2}{E}\,a^2 + 4\Gamma a\Big) = 0. \tag{3.3}$$

Using expression (2.6) for the attached mass of the crack $m(a)$, from (3.3) it is easy to obtain the expression

$$\frac{d}{dt}\left(a^2\,\frac{v^2}{v_{max}^2} - (a - a_{cr})^2\right) = 0. \tag{3.4}$$

Here

$$a_{cr} = \frac{2\Gamma E}{\pi(1-\nu^2)p_0^2}; \quad v_{max}(R) = \sqrt{\frac{1-\nu}{4(1+\nu)[(\nu^2 - 3\nu/2 + 7/8)\ln(2R/a) - 3\nu/8 + 3/32 + a^2/(8R^2)]}}\,\sqrt{\frac{E}{\rho}}. \tag{3.5}$$

The quantity a_{cr} has the meaning of the critical crack size [14] since the condition of positive acceleration dv/dt is satisfied only for cracks of half-length $a > a_{cr}$. Hence, only such cracks can move with acceleration in a homogeneous solid body. The quantity v_{max} is the limiting velocity of crack propagation with infinite increase in the crack size. The solution of Eq. (3.4) is written as

$$v = v_{max}(1 - a_{cr}/a).$$

In (3.5), v_{max} depends on the truncation radius of the integration region R, which is to be determined. The dependence of the maximum (limiting) velocity of v_{max} brittle fracture on R is logarithmic. The radius R is limited by the distance traveled by longitudinal elastic waves and linked to the velocity v_{max} and the longitudinal sound velocity c_l by the relation [2]

$$v_{max}(R) = c_l a/R. \tag{3.6}$$

In the case of brittle fraction considered, the truncation radius R is found from Eq. (3.6): $R = 2.6a$. We recall that the condition for the establishment of a quasi-equilibrium state is the inequality $v/c_l \ll 1$. Since $v/c_l \sim a/R \approx 1/2.6$, according to (3.6), this condition is poorly satisfied. However, the dependence of v_{max} on the truncation radius R is weak (logarithmic) (3.5), and, hence, one might expect that the obtained result differs only slightly from the exact value.

It should be noted that, according to [12], all results obtained in the present work also apply to the case of a plane stress state. In this case, in the above formulas, it is necessary to make the change

$$E \to E\Big(1 - \frac{\nu^2}{(1+\nu)^2}\Big), \qquad \nu \to \frac{\nu}{1+\nu}.$$

Figure 2 shows a curve of the limiting velocity versus Poisson's ratio for the cases of plane strain and plane stress.

Conclusions. The modeling of a crack by a continuous set of linear dislocations significantly simplified the problem of finding the potential and kinetic energies of the crack environment and also the attached mass of a moving crack. The indicated parameters of a moving crack are represented as double integrals which are easily calculated by changing the order of integration. As a result, the complete analytical expression for the limiting velocity $v_{\max}$ of supercritical cracks in homogeneous materials was obtained for the first time as a function of three characteristic parameters (E, ν, and ρ) of such materials (3.5).

The calculated value of $v_{\max}(R)$ turns out to depend on the truncation radius R of the integration region. However, the dependence $v_{\max}(R)$ is weak (logarithmic) (see 3.5), which allows one, by choosing the concrete value $R = 2.6a$ (a is the crack half-length) on the basis of solution of Eq. (3.6), to simplify the obtained analytical expression for $v_{\max}$:

$$v_{\max} = \sqrt{\frac{1-\nu}{4(1+\nu)[(\nu^2 - 3\nu/2 + 7/8)\ln 5.2 - 3\nu/8 + 3/32]}}\sqrt{\frac{E}{\rho}}.$$

Thus, it becomes possible to calculate the limiting velocity of crack propagation $v_{\max}$ in solid materials for which the main mechanical characteristics (E, ν, and ρ) are known. For example, for Poisson's ratio $\nu = 0.25$, the limiting crack velocity $v_{\max}$ in homogeneous materials calculated by formula (3.5) is equal to $v_{\max} = 0.38\sqrt{E/\rho}$, which coincides with the expression for $v_{\max}$ obtained numerically in [2].

We thank A. N. Mokhel' and R. L. Salganik for discussions of the work and useful remarks.

REFERENCES

1. N. F. Mott, "Fracture of metals: some theoretical considerations," *Engineering*, **165**, 16–18 (1948).
2. D. K. Roberts and M. A. Wells, "The velocity of brittle fracture," *Engineering*, **178**, No. 4639, 820 (1954).
3. V. M. Finkel', *Physics of Fracture* [in Russian], Metallurgiya, Moscow (1970).
4. Jin Yu, "The link between crack velocity and rupture time in creeping solids," *Eng. Fract. Mech.*, **53**, No. 2, 213–230 (1996).
5. P. A. Klein, J. W. Foulk, E. P. Chen, et al., "Physics-based modeling of brittle fracture: cohesive formulations and the application of meshfree methods," *Theor. Appl. Fract. Mech.*, **37**, 99–166 (2001).
6. K. B. Broberg, *Cracks and Fracture*, Academic Press, London (1999).
7. K. B. Broberg, "Constant velocity crack propagation — dependence on remote land," *Int. J. Solids Struct.*, **39**, 6403–6410 (2002).
8. T. Nishioka, "Computational dynamic fracture mechanics," *Int. J. Fract.*, **86**, 127–159 (1997).
9. K. Ravi-Chandar, "Dynamic fracture of nominally brittle materials," *Int. J. Fract.*, **90**, 83–102 (1998).
10. L. B. Freund, *Dynamic Fracture Mechanics*, Cambridge Univ. Press, Cambridge (1990).
11. G. I. Barenblatt, "Mathematical theory of equilibrium cracks in brittle fraction," *Zh. Prikl. Mekh. Tekh. Fiz.*, **4**, 3–56 (1961).
12. L. D. Landau and E. M. Lifshits, *Course of Theoretical Physics*, Vol. 7: *Theory of Elasticity*, Pergamon Press (1975).
13. I. I. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, Academic Press, New York (1980).
14. A. A. Griffith, "The phenomena of rupture and flow in solids, *Philos. Trans. Roy. Soc. London, Ser. A*, **221**, 163–198 (1920).